A Symmetrization of n-person Games

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Consider a symmetric zero-sum five-person game. If we modify this game by combining two players into one player, we have a four-person game which is symmetric with respect to three players. Von Neumann and Morgenstern made a detailed investigation into the connection between symmetric five-person games and four-person games with three symmetric players. It is the purpose of this short note to show that any constant-sum n-person game is equivalent to a game which is obtained from a symmetric constant-sum game, each group of players being considered as one player.

Since we shall confine our attention to constant-sum games, we shall mean a constant-sum game by a game. Let I be the set of all players in a n-person game, and for any subset R of I, let $-R$ be the complement of R and $|R|$ be the number of players belonging to R. Then the 0-1 normalized characteristic function $v(R)$ is the real-valued set function which satisfies the following conditions :

$$\begin{array}{llll} \text{(i)} & v(R)=0 & \text{if} & |R|=0 \quad \text{or} \quad 1, \\ & v(R)=1 & \text{if} & |R|=n \quad \text{or} \quad n-1, \\ \text{(ii)} & v(R)+v(-R)=1, & & \\ \text{(iii)} & v(R)+v(S) \leqq v(R \cup S) & \text{if} & R \cap S=\phi. \end{array}$$

For the sake of simplicity, we shall identify a game with its characteristic function or 0-1 normalized characteristic function.

Let Γ be a $(k+l)$-person game, which has k symmetric players

$$1, \quad 2, \quad \ldots, \quad k,$$

and l other players

$$a, \quad b_1, \quad \ldots, \quad b_{l-1}\,.$$

We shall say such a game has at most l degrees of asymmetry. Then it is sufficient for our purpose to show that there is a game Γ', which has at most $(l-1)$ degrees of asymmetry, and we can have a game, which is equivalent to Γ, from Γ' by combining some of players into one player.

If T denotes the set of players $\{b_1, \ldots, b_{l-1}\}$, value of the 0-1 normalized characteristic function of Γ is determined by the number of sym-

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metric players i and the subset S of $\{a\}\cup T$. So we may write as

(1) $\quad v(i, S)$.

By the condition (ii), if we know the values of (1) for $0\leqq i\leqq k$ and $S\subset T$ (the number of such values is $2^{l-1}(k+1)$), all other values are determined. But, since the condition (i) implies $(l+2)$ values

$$v(0, \phi)=v(1, \phi)=v(0, \{b_j\})=0 \qquad j=1, 2, \ldots, l-1,$$
$$v(k, T)=1,$$

a 0-1 normalized characteristic function is given by

(2) $\quad 2^{l-1}(k+1)-(l+2)$

values of (1). These values may be defined freely to some extent.

Now, let Γ' be a $((k+m)+(l-1))$-person game, which has $(k+m)$ symmetric players

$$1, 2, \ldots, k, k+1, \ldots, k+m,$$

and $(l-1)$ other players

$$b_1, \ldots, b_{l-1}.$$

Then Γ' has at most $(l-1)$ degrees of asymmetry, and its 0-1 normalized characteristic function is determined by

(3) $\quad 2^{l-2}(k+m+1)-(l+1)$

values. If we combine the players $k+1, \ldots, k+m$ into one player, we have a $(k+l)$-person game. Our purpose is to make this $(k+l)$-person game equivalent to Γ. If k, l, and m are fixed numbers, in order to construct a Γ' for any Γ, the relation

$$2^{l-2}(k+m+1)-(l+1)\geqq 2^{l-1}(k+1)-(l+2),$$

that is

$$m\geqq k+1-\frac{1}{2^{l-2}}$$

will be necessary. $m=k+1$ satisfies the inequality at all times. And Γ' becomes a $(2k+l)$-person game.

We hope to construct a $(2k+l)$-person game Γ' (or its characteristic function) which will answer our purpose for a given $(k+l)$-person game Γ. Γ is given by the values (1) for $0\leqq i\leqq k$ and $S\subset T$. Γ' is determined by values

$$v'(i, S) \qquad \text{for } 0\leqq i\leqq 2k+1,\ S\subset T.$$

But, for $k+1\leqq i\leqq 2k+1$, there exist relations

$$v'(i, S)=1-v'(2k+1-i, T-S)$$

and $\quad 2k+1-i \leqq k.$

So

(4) $\quad v'(i, S) \quad \text{for} \quad 0 \leqq i \leqq k, \ S \subset T$

are sufficient to determine Γ'.

We define

(5) $\quad v'(i, S) = (1-\alpha) v(i, S) \quad \text{for} \quad 0 \leqq i \leqq k, \ S \subset T,$

(6) $\quad v'(i, S) = 1 - v'(2k+1-i, T\text{-}S) \quad \text{for} \quad k+1 \leqq i \leqq 2k+1, \ S \subset T,$

where α is a number $2/3 \leqq \alpha < 1$ and v is the given $0-1$ normalized characteristic function.

We show v' satisfies the conditions (i), (ii) and (iii).

As to (i):

$$v(0, \phi) = v(1, \phi) = v(0, \{b_j\}) = 0 \qquad 1 \leqq j \leqq l-1$$

and (5) yield

$$v'(0, \phi) = v'(1, \phi) = v'(0, \{b_j\}) = 0. \qquad 1 \leqq j \leqq l-1$$

And

$$v'(2k, T) = 1 - v'(1, \phi) = 1,$$
$$v'(2k+1, T) = 1 - v'(0, \phi) = 1.$$

As to (ii): It is an immediate result of (6).

As to (iii): Let $S_1 \subset T$, $S_2 \subset T$ and $S_1 \cap S_2 = \phi$.

Case 1. If $0 \leqq i \leqq k$, $0 \leqq j \leqq k$ and $i+j \leqq k$,

$$\begin{aligned} v'(i, S_1) + v'(j, S_2) &= (1-\alpha)\{v(i, S_1) + v(j, S_2)\} \\ &\leqq (1-\alpha) v(i+j, S_1 \cup S_2) = v'(i+j, S_1 \cup S_2). \end{aligned}$$

Case 2. If $0 \leqq i \leqq k$, $k+1 \leqq j \leqq 2k+1$ and $k+1 \leqq i+j \leqq 2k+1$,

$$\begin{aligned} & v'(i+j, S_1 \cup S_2) - \{v'(i, S_1) + v'(j, S_2)\} \\ &= 1 - v'(2k+1-i-j, T - S_1 \cup S_2) - v'(i, S_1) \\ &\qquad - 1 + v'(2k+1-j, T - S_2) \\ &= v'(2k+1-j, T - S_2) - \{v'(2k+1-i-j, T - S_1 \cup S_2) \\ &\qquad + v'(i, S_1)\}. \end{aligned}$$

Here, $0 \leqq (2k+1-i-j) + i = 2k+1-j \leqq k$ and $(T - S_1 \cup S_2) \cap S_1 = \phi$.

So, by the result of the case 1,

$$v'(2k+1-i-j, T - S_1 \cup S_2) + v'(i, S_1) \leqq v'(2k+1-j, T - S_2).$$

Hence,

$$v'(i+j, S_1 \cup S_2) - \{v'(i, S_1) + v'(j, S_2)\} \geqq 0.$$

Case 3. If $0 \leqq i \leqq k$, $0 \leqq j \leqq k$ and $k+1 \leqq i+j \leqq 2k+1$,

$$\begin{aligned} & v'(i+j, S_1 \cup S_2) - \{v'(i, S_1) + v'(j, S_2)\} \\ &= 1 - (1-\alpha)\{v(2k+1-i-j, T - S_1 \cup S_2) \\ &\qquad + v(i, S_1) + v(j, S_2)\} \end{aligned}$$

$$\geqq 1-(1-\alpha)3 \geqq 0,$$

since all the values of v are $0 \leqq v \leqq 1$ and $2/3 \leqq \alpha \leqq 1$.

Now, we get a game $\bar{\Gamma}$ by combining the players $k+1, \ldots, 2k+1$ of Γ' into one player named a. We show that $\bar{\Gamma}$ is equivalent to Γ.

The set of players of $\bar{\Gamma}$ consists of $1, 2, \ldots, k$, a and T, and the characteristic function $\bar{v}$ of $\bar{\Gamma}$ is defined as follows : the value for the subset of players not containing the player a is

$$\bar{v}(i, \mathrm{S}) = v'(i, \mathrm{S}) \quad \text{for} \quad 0 \leqq i \leqq k, \quad \mathrm{S} \subset \mathrm{T},$$

where i denotes the number of players picked up from $1, 2, \ldots, k$. And if the subset contains the player a

$$\bar{v}(i, \{a\} \cup \mathrm{S}) = v'(j, \mathrm{S}) \qquad \text{where} \qquad j = k+1+i, \quad \mathrm{S} \subset \mathrm{T}.$$

The function $\bar{v}$ is not $0-1$ normalized, since

$$\bar{v}(0, \{a\}) = v'(k+1, \phi) = 1 - v'(k, \mathrm{T})$$
$$= 1 - (1-\alpha) v(k, \mathrm{T}) = \alpha \neq 0$$

The $0-1$ normalised characteristic function $\tilde{v}$ of $\bar{v}$ is obtained by

$$\tilde{v}(i, \mathrm{S}) = \frac{v(i, \mathrm{S})}{1 - v(0, \{a\})} = \frac{v'(i, \mathrm{S})}{1-\alpha} = v(i, \mathrm{S})$$

And this means $\hat{v} = v$ and hence $\bar{\Gamma}$ is equivalent to Γ.